STOCHASTIC FRACTIONAL HP EQUATIONS

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Abstract: In this paper we established the condition for a curve to satisfy stochastic fractional HP (Hamilton-Pontryagin) equations. These equations are described using Itô integral. We have also considered the case of stochastic fractional Hamiltonian equations, for a hyperregular Lagrange function. From the stochastic fractional Hamiltonian equations, Langevin fractional equations were found and numerical simulations were done.

Keywords: HP equations, stochastic fractional equations, stochastic flows, hyperregular function, fractional Langevin equations, Euler scheme.

1 Introduction

J.M. Bismut was the first one that introduced concepts of stochastic geometric mechanics, in his work from 1981, when he defined the notion of "stochastic Hamiltonian system". He showed that that the stochastic flow of a certain randomly perturbed Hamiltonian systems on flat spaces extremizes a stochastic action, and using this property, he proved symplecticity and Noether theorem for stochastic Hamiltonian systems. Since then, there has been a need in finding out tools and algorithms for the study of this kind of systems with uncertainty. Bismut's work was continued by Lazaro-Cami and Ortega ([17], [18]), in the sense that his work was generalized to manifolds, stochastic Hamiltonian systems on manifolds extremize a stochastic action on the space of manifold valued semimartingales, the reduction of stochastic Hamiltonian system on cotangent bundle of a Lie group, a counter example for the converse of Bismut's original theorem.

Very important in many science fields is fractional calculus: fractional derivatives, fractional integrals, of any real or complex order. Fractional calculus is used when fractional integration is needed. It is used for studying simple dynamical systems, but it also describes complex physical systems. For example, applications of the fractional calculus can be found in chaotic dynamics, control theory, stochastic modelling, but also in finance, hydrology, biophysics, physics, astrophysics, cosmology and so on ([6], [9], [10]). But some other fields have just started to study problems from fractional point of view. In great fashion is the study of fractional problems of the calculus of variations and Euler-Lagrange type equations. There were found Euler-Lagrange equations with

fractional derivatives, and then Klimek found Euler-Lagrange equations, but with symmetric fractional derivatives [16]. Most famous fractional integral are Riemann-Liouville, Caputo, Grunwald-Letnikov and most frequently used is Riemann-Liouville fractional integral. The study of Euler-Lagrange fractional equations was continued by Agrawal [2] that described these equations using the left, respectively right fractional derivatives in the Riemann-Liouville sense. This fractional calculus has some great problems, such as presence of non-local fractional differential operators, or the adjoint fractional operator that describes the dynamics is not the negative of itself, or mathematical calculus may be very hard because of the complicated Leibniz rule, or the absence of chain rule, and so on. After O.P. Agrawal's formulation [2] of Euler-Lagrange fractional equations, Băleanu and Avkar [4] used them in formulating problems with Lagrangians linear in velocities. Standard multi-variable variational calculus has also some limitations. But in [23] C. Udrişte and D. Opriş showed that these limitations can by broken using the multi-linear control theory.

For fractional stochastic integrals of the form

$$X_t = X_0 + \int_0^t F(t,s,X_s)ds + \int_0^t G_a(H_t;t,s,X_s)dW_s^a, \tag{1}$$

the existence and uniqueness of its solution was discussed by Pardoux and Protter in their work [20].

In [9], it was proved the existence, uniqueness, and continuity of a fractional stochastic equation of the form

$$X_t = X_0 + I_t^\beta F(t,s,X_s) + W_t^\beta G^a(H_t;t,s,X_s), \tag{2}$$

where $0<\beta<1$, $I_t^\beta F(t,s,X_s)$. The fractional integral of $F(t,s,X_s)$ is defined by

$$I_t^\beta F(t,s,X_s) = \frac{1}{\Gamma(\beta)}\int_0^t \frac{F(t,s,X_s)}{(t-s)^{1-\beta}}ds,$$

and fractional Wiener process $W_t^\beta G_a(H_t;t,s,X_s)$ of $G_a(H_t;t,s,X_s)$ is

$$W_t^\beta G_a(H_t;t,s,X_s) = \frac{1}{\Gamma(\frac{\beta+1}{2})}\int_0^t \frac{G_a(H_t;t,s,X_s)}{(t-s)^{(1-\beta)/2}}dW_s^a.$$

One application of fractional stochastic equation of the form (2) is in finance. Fractional Black-Scholes market is described in terms of the bank account and a stock. The price at a time t is given by the following formula

$$A_t = exp\Big(\int_0^t r(s)ds\Big), \tag{3}$$

where $r(s)\geq 0$, $s\in[0,t]$, represents the interest rate. The price can be expressed using a fractional Volterra-type equation:

$$X_t = X_0 + \frac{1}{\Gamma(\beta)}\int_0^t \frac{\mu(s)X_s}{(t-s)^{1-\beta}}ds + \frac{1}{\Gamma(\frac{\beta+1}{2})}\int_0^t \frac{\sigma(s)X_s}{(t-s)^{(1-\beta)/2}}dW_s, \tag{4}$$

where $\mu,\sigma\geq 0$ are continuous functions on $[0,T]$.

In this paper, we restrict our attention to stochastic fractional Hamiltonian systems characterized by Wiener processes and assume that the space of admissible curves in configuration space is of class C^1. Random effects appear in the balance of momentum equations, as white noise, that is why we may consider randomly perturbed mechanical systems. It should be mentioned that the ideas in this paper can be readily extended to stochastic Hamiltonian systems [19] driven by more general semimartingales, but for the sake of clarity we restrict to Wiener processes. Within this context, the results of the paper are as follows:

1. The paper proves almost surely that a curve satisfies stochastic fractional HP equations if and only if it extremizes a stochastic action. This theorem is the main result of the paper;

2. Fractional HP equations are described using fractional Riemann-Liouville integral and fractional Itô integral;

3. Langevian type stochastic fractional equations are obtained, in the case of a hyper-regular Lagrange function.

The paper is organized as follows. In Section 2 we present some sufficient conditions for existence, uniqueness and almost sure differentiability of stochastic flows on manifolds. In Section 3, we extend the fractional Hamilton-Pontryagin (HP) principle to the stochastic setting to prove that a class of mechanical systems with multiplicative noise appearing as forces and torques possess a variational structure. In Section 4, for a hyperregular Lagrange function, we get the stochastic fractional Hamiltonian equations that lead to Langevin fractional equations. For a fractional Lagrange function, defined on $\mathbb{R}^2$, the corresponding fractional Langevin equations are simulated. The mechanical system could evolve on a nonlinear configuration space and involve holonomic constraints or nonconservative effects in the drift. The fractional Hamiltonian and the Lagrangian description are joined together to get the fractional HP system.

2 Stochastic flows on manifolds

Some standard results on flows of SDE on manifolds are reviewed here to the reader's convenience. For more detailed exposition, the reader is referred to the textbooks such as [12] or [15]. This section parallels the treatment of deterministic flows on manifolds found in [1].

Let M be a manifold modelled on a Banch space E. Recall that a vector field on the manifold M is a section of the tangent bundle TM on M. The set of all C^k vector fields on M is denoted by $\mathcal{X}^k(M)$.

A stochastic dynamical system consists of a base flow on the probability space which propagates the noise, and a stochastic flow on M which depends on the noise.

A stochastic dynamical system consists of a base flow on the probability space $(\Omega, \mathcal{F}, P)$ and a stochastic flow on a manifold M. The *base flow* is a $P-$preserving map $\theta : \mathbb{R}\times\Omega \to \Omega$ which satisfies:

1. $\theta_0 = id_\Omega : \Omega \to \Omega$ is the identity on Ω;

2. for all $s, t \in \mathbb{R}$, the group property,$\theta_s \circ \theta_t = \theta_{t+s}$.

Given times $0 \leq r \leq s \leq t$, the "stochastic flow" on M is a map $\varphi_{t,s} : \Omega \times M \to M$ such that

1. for almost all $\omega \in \Omega$, the map $(s,t,\omega,x) \mapsto \varphi_{t,s}(\omega)x$ is continuous in s,t and x;

2. $\varphi_{s,s}(\omega) = id_M : M \to M$ is the identity map on M, for all $s \in \mathbb{R}$;

3. φ satisfies the cocycle property

$$\varphi_{t,s}(\theta_s(\omega)) \circ \varphi_{s,r}(\omega) = \varphi_{t,r}(\omega).$$

This paper is concern with stochastic dynamical systems that come from stochastic laws of motion, i.e. ones whose stochastic flows define solutions of stochastic differential equations. Consider a manifold M, modelled on a Banach space E and vector fields $X_0, X_a \in \mathfrak{X}^k(M)$, $a = 1,...,m$. Let $(W^a(t,\omega), \mathcal{F}_\tau)$, $a = 1,...,m$, be independent Wiener processes for $0 \le t \le T$. In terms of these objects, the Stratonovich stochastic differential equations, that the paper considers, takes the form:

$$dx = X_0(x)dt + X_a(x) \circ dW^a,\ x(0) = x_0. \tag{5}$$

A *Stratonovich integral curve* of (5) is a C^0-map, $c(\cdot,\omega) : [0,T] \to M$ which satisfies

$$c(t,\omega) = x_0 + \int_0^t X_0(c(s,\omega))ds + \int_0^t X_a(c(s,\omega)) \circ dW^a(s,\omega)ds,$$

for all $t \in [0,T]$.

Let c be a Stratonovich integral curve of (5). *Pathwise uniqueness* of c means that if $\bar{c} : I \to M$ is also a solution of (5) on the same filtered probability space, with the same Brownian motion and initial random variables, then,

$$P(c(t,\omega) = \bar{c}(t,\omega), \quad \forall\, t \in [0,T]) = 1.$$

For the rest of the paper the explicit dependence of stochastic maps on the point $\omega \in \Omega$ will usually be suppressed. With these definitions, one can state the following key, but standard theorem ([11], [12], [15]).

Theorem 1 *(Existence, uniqueness and smoothness)*
Let M be a manifold with the model space E. Suppose that $X_0, X_a \in \mathfrak{X}^k(M)$, $a = 1,...,m$ and $k \ge 1$ are uniformly Lipschitz and measurable with respect to $x \in M$. Let $I = [0,T]$. Then the following statements hold.

1. *For each $u \in M$, there is almost surely a C^0-curve, $c : I \to M$, such that $c(0) = u$ and c satisfies (5), for all t. This curve $c : I \to M$ is called a* maximal solution*;*

2. *c is pathwise unique;*

3. *There is almost surely a mapping $F : I \times M \to M$, such that the curve $c_u : I \to M$, defined by $c_u(t) = F_t(u)$, is a curve satisfying (5), for all $t \in I$. Moreover, almost surely F is C^k in u and C^0 in t.*

□

3 Stochastic fractional HP mechanics

In this section, a fractional variational principle is introduced for a class of stochastic fractional Hamiltonian systems on manifolds. The stochastic fractional action is a sum of classical fractional action and several stochastic integrals.

Let Q be an $n-$dimensional manifold, $\gamma_a : Q \to \mathbb{R}$, $a = 1, ..., m$, a deterministic function, and the Lagrangian $L : TQ \to \mathbb{R}$. Let $(\Omega, \mathcal{F}, P)$ be a probability space and the interval $[a, b] \subset \mathbb{R}$. Let $\{W^a(t), \mathcal{F}_t\}_{t\in[a,b]}$, for $a = 1, ..., m$, where $\{W^a\}_{a=1,...,m}$ are independent, real-valued Wiener processes and $\mathcal{F}_t$ is the filtration generated by these Wiener Processes.

The stochastic HP fractional action is defined by $\mathcal{A}^{\alpha,\beta} : \Omega \times \mathcal{C}(PQ) \to \mathbb{R}$:

$$\mathcal{A}^{\alpha,\beta}(q, v, p, t) = \frac{1}{\Gamma(\alpha)} \int_a^b (L(q(s), v(s))(t-s)^{\alpha-1} + (p(s), \dot{q}(s) - v(s))(t-s)^{\alpha-1})ds+$$
$$+\frac{1}{\Gamma(\beta)} \int_a^b \gamma_a(q(s))(t-s)^{\beta-1} \circ dW^a(ds), \tag{6}$$

where $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $\Gamma(\alpha)$, $\Gamma(\beta)$ are the Euler gamma functions and $PQ = TQ \oplus T^*Q$, and

$$\mathcal{C}(PQ) = \{(q, v, p) \in \mathrm{C}^0([a, b], PQ) | q \in \mathrm{C}^1([a, b], Q),\ q(a) = q_a,\ q(b) = q_b\}. \tag{7}$$

The first integral in (6) is a Riemann integral, and the second one is an Itô integral.

The HP path space is a smooth infinite-dimensional manifold. One can show that its tangent space in $c = (q, v, p) \in \mathcal{C}([a, b], q_1, q_2)$ consists of maps $w = (q, v, p, \delta q, \delta v, \delta p) \in \mathrm{C}^0([a, b], T(PQ))$, such that $\delta q_a = \delta q_b = 0$, and q, δq are of class C^1. Let us denote by $(q, v, p)(\cdot, \varepsilon) \in \mathcal{C}(PQ)$ the one-parameter family of curves in $\mathcal{C}$, that is differentiable with respect to ε. Define the differential of $\mathcal{A}^{\alpha,\beta}$ as

$$d\mathcal{A}^{\alpha,\beta}(\delta q, \delta v, \delta p) = \frac{\partial}{\partial \varepsilon} \mathcal{A}^{\alpha,\beta}(\omega, q(s, \varepsilon), v(s, \varepsilon), p(s, \varepsilon))\Big|_{\varepsilon=0}, \tag{8}$$

where

$$\delta q(s) = \frac{\partial}{\partial \varepsilon} q(s, \varepsilon)\Big|_{\varepsilon=0},\ \delta q(a) = \delta q(b) = 0,\ \delta v(s) \frac{\partial}{\partial \varepsilon} v(s, \varepsilon)\Big|_{\varepsilon=0},\ dp(s) = \frac{\partial}{\partial \varepsilon} p(s, \varepsilon)\Big|_{\varepsilon=0}. \tag{9}$$

In terms of this differential, one can state the following critical point condition for the action $\mathcal{A}^{\alpha,\beta}$.

Theorem 2 *(Stochastic fractional variational principle of HP)*
Let $L : TQ \to \mathbb{R}$ be a Lagrangian on TQ of class C^2, with respect to q and v, and with globally Lipschitz first derivative with respect to q and v. Let $\gamma_a : Q \to \mathbb{R}$ be of class C^2 and with globally Lipschitz first derivatives, for $a = 1, ..., m$. Then, almost surely, a curve $c = (q, v, s) \in \mathcal{C}(PQ)$ satisties the stochastic fractional equations

$$dq(s) = v(s)ds,$$
$$dp(s) = \left(\frac{\partial L}{\partial q}(q(s), v(s)) + \frac{\partial L}{\partial v}(q(s), v(s))\frac{\alpha-1}{t-s}\right)ds + \frac{\partial \gamma_a(q(s))}{\partial q}\frac{\Gamma(\alpha)}{\Gamma(\beta)}(t-s)^{\beta-\alpha} \circ dW^a(s),$$
$$p(s) = \frac{\partial L}{\partial v}(q(s), v(s)),\ s \in [a, b], \tag{10}$$

if and only if it is a critical point of the function $\mathcal{A}^{\alpha,\beta} : \Omega \times \mathcal{C}(PQ) \to \mathbb{R}$, i.e. $d\mathcal{A}^{\alpha,\beta}(c) = 0$.

Proof: The proof results by applying the method from [5] and [10]. □

Observe that by the Itô-Stratonovich conversion formula, the Itô modification to the drift is equal to 0, and hence (10) can be written in the Itô form as

$$dq = vds,$$

$$dp = \left(\frac{\partial L}{\partial q}(q,v) + \frac{\partial L}{\partial v}(q,v)\frac{\alpha-1}{t-s}\right)ds + \frac{\partial \gamma_a(q)}{\partial q}\frac{\Gamma(\alpha)}{\Gamma(\beta)}(t-s)^{\beta-\alpha}dW^a(s), \tag{11}$$

$$p = \frac{\partial L}{\partial v}(q,v),\ s \in [a,b].$$

In what follows, structure-preserving properties of the flow map, defined by maximal solution of the equations over $[a,b]$ will be investigated. First, observe that because of smoothness conditions assumed in Theorem 2, a solution almost surely exists and it is pathwise unique on $[a,b]$, by the result from Section 2. When γ_a is constant for, $a = 1, ..., m$ and $\alpha = 1$, the reader is referred to [24], for deterministic treatments of symplecticity, momentum map preservation and holonomically constrained mechanical systems.

If $\beta = \alpha$, the equation (11) is given by:

$$dq = vds,$$

$$dp = \left(\frac{\partial L}{\partial q}(q,v) + \frac{\partial L}{\partial v}(q,v)\frac{\alpha-1}{t-s}\right)ds + \frac{\partial \gamma_a(q)}{\partial q}dW^a(s), \tag{12}$$

$$p = \frac{\partial L}{\partial v}(q,v).$$

If γ_a is constant, for $a = 1, ..., m$, from (11), results:

$$dq = vds,$$

$$dp = \left(\frac{\partial L}{\partial q}(q,v) + \frac{\partial L}{\partial v}(q,v)\frac{\alpha-1}{t-s}\right)ds, \tag{13}$$

$$p = \frac{\partial L}{\partial v}(q,v),$$

and they represent the Euler-Lagrange fractional equations ([10], [13]).

4 Stochastic fractional equation for the Lagrangian hyperregular

Let $L : TQ \to \mathbb{R}$ be a Lagrangian on TQ hyperregular, that means $det\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right) \neq 0$. From (10) results the following propositions:

Proposition 3 *(Stochastic fractional Hamiltonian equations)*
The equations (11) *are equivalent with the equations:*

$$dq^i = \frac{\partial H}{\partial p_i}ds,$$

$$dp_i = \left(-\frac{\partial H}{\partial q^i} + \frac{\alpha-1}{t-s}p_i\right)ds + \frac{\Gamma(\alpha)}{\Gamma(\beta)}\frac{\partial \gamma_a(q)}{\partial q^i}(t-s)^{\beta-\alpha}dW^a(s),\ i = 1, ..., n, \tag{14}$$

where $H = p_i q^i - L(q, v)$. $\square$

Proposition 4 *If* $L = \frac{1}{2} g_{ij} v^i v^j$, *where* g_{ij} *are the components of a metric on the manifold* Q, *equations* (11) *take the form:*

$$\begin{aligned}
&dq^i = v^i ds,\\
&dv^i = -\Big(\Gamma^i_{jk} v^j v^k + \tfrac{\alpha-1}{t-s} v^i\Big) ds + \tfrac{\Gamma(\beta)}{\Gamma(\alpha)} g^{ij} \tfrac{\partial \gamma_a(q)}{\partial q^j} (t-s)^{\beta-1} dW^a(s),\ i, j = 1, ..., n,
\end{aligned} \tag{15}$$

where Γ^i_{jk} *are Cristofel coefficients associated to the considered metric. Equations* (14) *become:*

$$\begin{aligned}
&dq^i = v^i ds,\\
&dp_i = \Big(\tfrac{1}{2} \tfrac{\partial g_{kl}}{\partial q^i} p^k p^l + \tfrac{\alpha-1}{t-s} p_i\Big) ds + \tfrac{\Gamma(\alpha)}{\Gamma(\beta)} \tfrac{\partial \gamma_a(s)}{\partial q^i} (t-s)^{\beta-\alpha} dW^a(s),\ i, j, k = 1, ..., n,
\end{aligned} \tag{16}$$

$\square$

Equations (14) represent fractional Langevin equations. Equations (16) can be used for fractional motion of relativistic particles with noise.

Proposition 5 **a)** *If* $Q = \mathbb{R}$, $H(p, q) = \frac{1}{2} p^2 + U(q)$ *and* $\gamma(q) = \cos(q)$, *equations* (14) *are given by:*

$$\begin{aligned}
&dq = p ds,\\
&dp = \Big(-\tfrac{dU}{dq} + \tfrac{\alpha-1}{t-s} p\Big) ds - \tfrac{\Gamma(\alpha)}{\Gamma(\beta)} (t-s)^{\beta-\alpha} \sin(q) dW(s);
\end{aligned} \tag{17}$$

b) *If* $U(q) = \cos(q)$, *the Euler scheme for* (17)*is:*

$$\begin{aligned}
&x(n+1) = x(n) + h y(n),\\
&y(n+1) = y(n) + h(\sin(x(n)) + \tfrac{\alpha-1}{t-nh} y(n)) - \tfrac{\Gamma(\alpha)}{\Gamma(\beta)} (t-nh)^{\beta-\alpha} sin(x(n)) G(n),
\end{aligned} \tag{18}$$

where $h = \frac{T}{N}$, $G(n) = W((n+1)h) - W(nh)$, $n = 0, ..., N-1$, $0 < \alpha < 1$, $0 < \beta < 1$, *and* $x(n) = q(nh)$, $y(n) = p(nh)$.

For $\alpha = 0.6$, $\beta = 0.3$, $t = 0.8$ and $h = 0.0001$, with Maple 13, the orbit $(n, p(nh))$ is represented in Figure 1, and the orbit $(n, p(nh, \omega))$ is represented in Figure 2

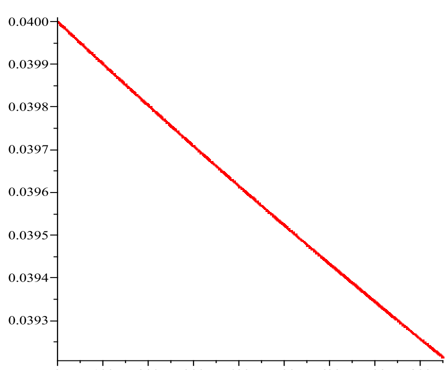

Figure 1: the orbit $(n, p(nh))$

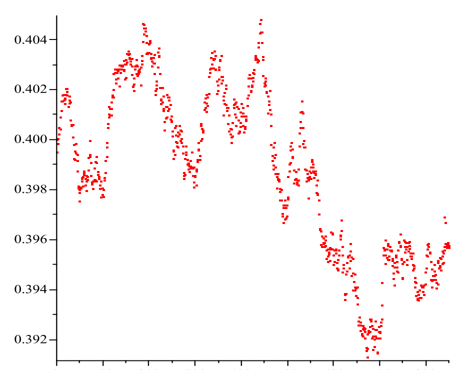

Figure 2: the orbit $(n, p(n, \omega))$

In Figure 3 the orbit $(q(nh), p(nh))$ is represented, and in Figure 4 it is represented the orbit $(q(nh, \omega), p(nh, \omega))$.

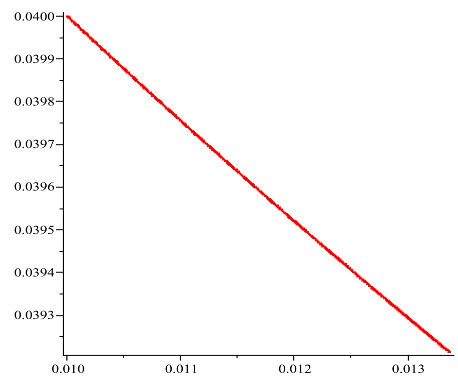

Figure 3: the orbit $(q(nh), p(nh))$

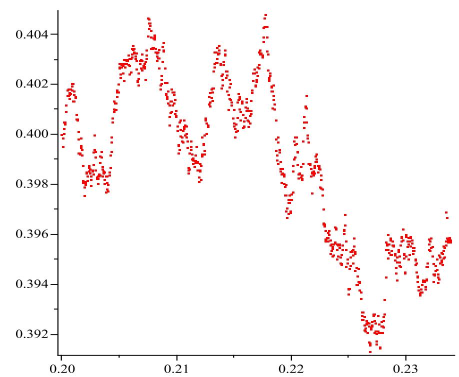

Figure 4: the orbit $(q(nh, \omega), p(nh, \omega))$

For $\alpha = \beta = 0.6$, and $t = 0.8$, $h = 0.0001$, the orbits $(n, p(nh, \omega))$ and $(q(nh, \omega), p(nh, \omega))$ are represented in Figure 5 and Figure 6.

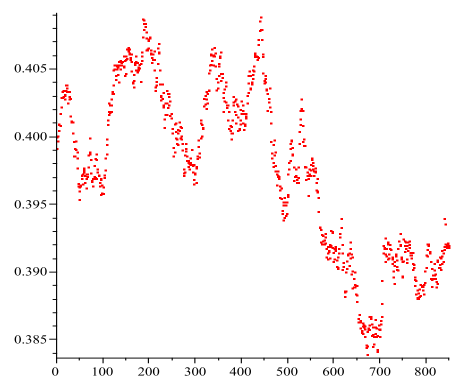

Figure 5: the orbit $(n, p(nh, \omega))$

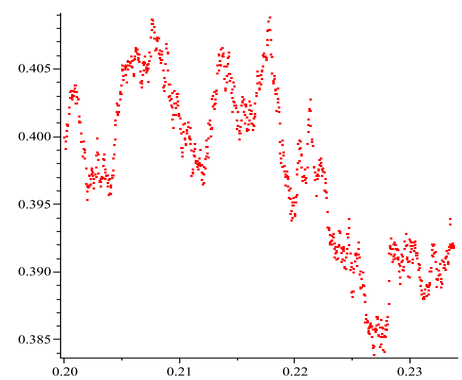

Figure 6: the orbit $(q(nh, \omega), p(nh, \omega))$

Conclusions

In this paper it was described stochastic fractional HP principle, using classical stochastic HP principle [5] and fractional principle ([10], [13]). Using a hyperregular Lagrange function, Langevin-type fractional equations were illustrated. We have done the numerical simulation for the case of a Lagrange function defined on $\mathbb{R}^2$.

In the future work, we will consider other problems that deal with stochastic fractional HP principle.

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